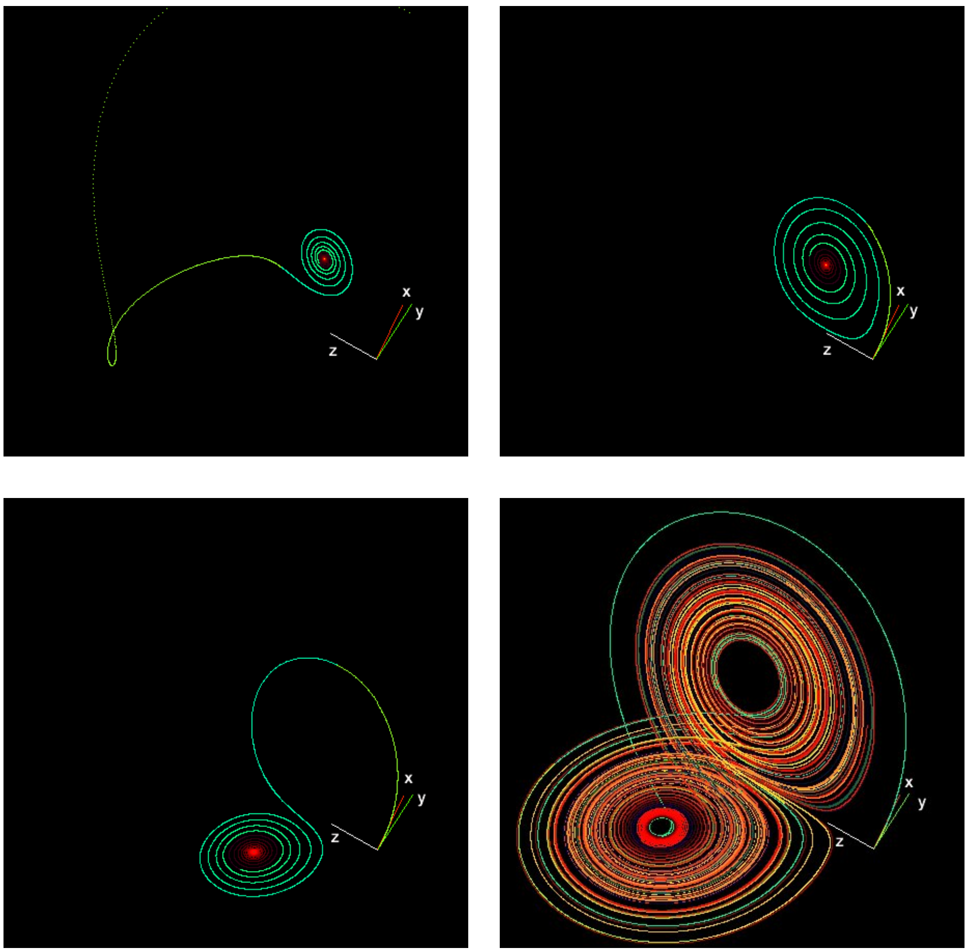
The Poincaré – Bendixson Theorem



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Part 1: The Poincaré – Bendixson Theorem

1.1) State the theorem and Explain in simple terms what it means

- Use diagrams if needed
- Give examples of simple Dynamic Systems
    - that fulfill the hypotheses of the theorem
    - that fail to fulfill the hypotheses of the theorem
- If possible separate 1D and 2D case

1.2) Explain why it does not hold in 3D and above?

- use simple terms
- diagrams welcome

Part 2 : Showing the trajectories are bounded

To apply Poincaré – Bendixson, we need to

- Study the steady points of the Dynamic System (see WIKI Document 1)
  As we know it is a tedious task, but a pretty straightforward one as ii implies studying the sign of the trace and determinant of the Jacobian matrix.

- Prove that the trajectories remain bounded with time.
  As we shall see this is in general tricky. No general method exists.

We detail in this section the three most common approaches to show that a trajectory of a Dynamic System remains bounded.

2.1) Finding the Equation of the Trajectory

The typical example is the Lotka-Volterra model. Finding the equation of the trajectory is achieved as follows

- Eliminate the time differential dt in order to obtain the differential form

$$f(u,v)du + g(u,v)dv = 0$$

- Integrate (if you can)

$$F(u,v) = 0$$

Now all you have left to do is to prove that the corresponding curve is bounded in the plane, that is that there is no way to infinity along the curve.

Example: Consider the Lotka - Volterra model

$$\text{(Lotka – Volterra)} \qquad \begin{cases} \dfrac{dX}{dt} = aX - bXY \\ \dfrac{dY}{dt} = cXY - dY \end{cases}$$

Show that the curves are bounded (plotting them is a bonus)

2.2) Comparison with another Dynamic System

The main idea is simple.

First Step: We introduce another (well chosen) dynamic system whose trajectories are bounded. This new dynamic system of reference has its own set of parameters (possibly a different number and in general with a different meaning).

Second Step: Then we prove that for any initial condition (X_0, Y_0) the trajectory governed by our system of interest remains within the trajectory of the second better-behaved system.

NB: The method still works if the trajectories of the dynamic system of reference are bounded only for a subset of the space of parameters. What matters is that when we do the comparison the second trajectory is bounded -we therefore need to make sure that we place ourselves in the correct subset of parameters.

In practice <u>the realisation is as hard and messy as the idea is simple!!</u>

Indeed we need to find a dynamic system that will compare nicely with the system we study (this requires imagination, experience and luck). Furthermore even when we have chosen well the dynamic system, the comparison is often tedious and carred out piecewise (i.e one piece of space after another).

Example: Consider the simple following Dynamic System (another Prey-Predator model)

(Prey – Predator 2)
$$\begin{cases} \dfrac{dX}{dt} = \dfrac{aX}{a_0 + X} - bXY \\ \dfrac{dY}{dt} = cXY - dY \end{cases}$$

The growth of the predator is bounded. We can prove simply that it has 2 steady points (0,0) which is a saddle point and another one (find it!!) that is stable. The flow chart reminds a lot of that of Lotka-Volterra's (the trajectories spin around the second steady point).
All this suggests that comparison with Lotka-Volterra will be fruitful.

Carry on the analysis and prove that indeed the trajectories of the second model are trapped inside the (closed) trajectories of Lotka-Volterra for well-chosen parameters.

Hint: to find the correct parameters , make sure that the steady point of both models coincide.

2.3) Using Lyapunov Functions

2.3.a) Give a brief theoretical introduction on Lyapunov functions

In particular focus on the following uses :

- How can they be used to prove the stability of a steady point? If possible give an example of your own making
- How can they be used to show that trajectories are bounded?

Note: Finding Lyapunov functions is an art form! There are some functions that work for few well-known families of systems. In general it takes times, luck, intuition…

2.3.b) Example

We are going to consider the following (complex) system

(Prey – Predator 3)

$$\begin{cases} \dfrac{dX}{dt} = \dfrac{aX}{a_0 + X} - b\dfrac{XY}{b_0 + X} - eX \\ \dfrac{dY}{dt} = c\dfrac{XY}{c_0 + XY} - dY \end{cases}$$

Its dynamic is very complex but it is very easy to prove that for d and e strictly positive the trajectories remain bounded!!

We only consider positive trajectories $\left(X(t) > 0, Y(t) > 0\right)$. It is easy to show that such trajectories correspond to strictly positive initial conditions $\left(X_0 > 0, Y_0 > 0\right)$.

$F(X,Y) = X + Y$ is therefore a Lyapunov function for them. Show the following results

- $\dfrac{dF(X,Y)}{dt} = \dfrac{dX}{dt} + \dfrac{dY}{dt} < 0$ for points far away from the origin (you do not have to estimate the distance accurately, just show that there is such a distance).

- All trajectories of the system are bounded (please do the reasoning manually and do not use the criterion spelt in 2.3.c)